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The dynamic contact of an impactor and an elastic orthotropic plate when there are propagating thermoelastic waves☆

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ABSTRACT

The onset and propagation of longitudinal and transverse elastic waves and a temperature wave in a circular plate possessing cylindrical anisotropy is investigated. As a result of this the plate becomes deformed when there is a temperature shock or shock interaction with the heated body. The plate is assumed to be fairly extended and reflected waves are ignored when calculating the stress state. The onset of a temperature wave is possible if the non-classical hyperbolic-type heat-conduction equation is employed. To determine the displacements of points outside the contact region a ray method is employed based on the use of power series with respect to time and a spatial coordinate as well as compatibility conditions. The displacements and internal forces are determined in the form of sections of ray series apart from integration constants, which are found from the contact conditions. The temperature effect on the propagating wave is investigated in the form of second-order discontinuity surfaces.

Problems of the impact of a thermoelastic rod against a thermally insulated side surface on a heated massive obstacle were considered for a finite heat of propagation velocity in Refs. 1,2, in which the Laplace transformation method was employed and the diffusion term in the heatconduction equation was ignored, i.e., it was assumed that the temperature field in the rod is totally wave-like.[1] An expansion in power series, similar to that proposed earlier,[3] was used to solve this problem.[2] The propagation of thermoelastic waves was only taken into account in the impactor.

An expression for the heat flux, propagating with a finite velocity, was obtained for the first time in Ref. 4. The dependence of the temperature on the distance between the point of delivery of the heat and a specified location was found using a numerical method, based on the boundary elements method, a Laplace transformation and the hyperbolic heat-conduction equation.[5] In everyday engineering problems, related to short-term processes, high temperature density and certain other phenomena, it is often necessary to take into account dependences of the temperature, time and quantity of heat, passing through the cross section, which differ from the Fourier relation.

Using the wave equations for an orthotropic plate,[6] the ray method and the hyperbolic law of heat conduction, we obtain the temperature field and the parameters of a thermoelastic wave, and we investigate its effect on the dynamic characteristics of the contact.

1. Fundamental equations

The displacements of points of an elastic orthotropic plate, possessing cylindrical anisotropy, when there is inertia of rotation, deformation of the shear of the transverse sections and thermoelastic deformations are found from the equations[6]

$$D_r\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)-D_\theta\frac{\varphi}{r^2}+hKG_{rz}\left(\frac{\partial w}{\partial r}-\varphi\right) = -\rho\frac{h^3}{12}\frac{\partial^2\varphi}{\partial t^2}+\frac{2(1+\nu_r)}{1-2\nu_r}\frac{\partial(\alpha_1 T)}{\partial r} \tag{1.1}$$

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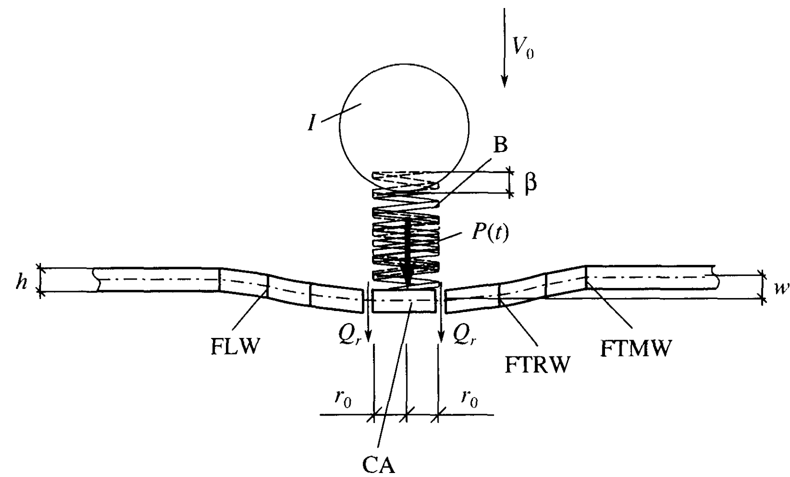

Fig. 1.

$$KG_{rz}\left(\frac{\partial^2 w}{\partial r^2} - \frac{\partial \varphi}{\partial r}\right) + KG_{rz}\frac{1}{r}\left(\frac{\partial w}{\partial r} - \varphi\right) = \rho\frac{\partial^2 w}{\partial t^2} \quad (1.2)$$

where

$$D_r = \frac{h^3}{12}B_r, \quad D_\theta = \frac{h^3}{12}B_\theta, \quad B_r = \frac{E_r}{1-\nu_r\nu_\theta}, \quad B_\theta = \frac{E_\theta}{1-\nu_r\nu_\theta}, \quad E_r\nu_r = E_\theta\nu_\theta, \quad K = \frac{5}{6}$$

D_r and D_θ are the bending stiffnesses for the directions r and θ respectively, E_r and E_θ and ν_r and ν_θ are the moduli of elasticity and Poisson's ratios for the directions r and θ, G_{rz} is the shear modulus in the rz plane, $w(r, \theta)$ is the normal displacement of the middle plane, $\varphi(r, \theta)$ is the angle of rotation of the normal filament outside the contact area, ρ is the density, h is the plate thickness, t is the temperature and α_1 is the coefficient of thermal expansion of the material.

We will consider the hyperbolic-type heat conduction equation[5]

$$\frac{\partial T}{\partial t} + \tau_q\frac{\partial^2 T}{\partial t^2} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \quad (1.3)$$

where $\alpha = k/(\rho c)$, k is the heat transfer coefficient between the plate and the medium at the heating point, c is the specific heat capacity and τ_q is the time delay in establishing the heat flux.

2. The ray method

When there is an instantaneous heat source of or after a heated body collides with a plate a contact area of radius r_0 is formed in the plate and longitudinal, shear and thermoelastic waves are generated (Fig. 1), the wave fronts of which are cylindrical surfaces of second-order discontinuity, which spread with a normal velocity $G^{(\gamma)}$ ($\gamma = 1, 2, 3$ and denotes the number of the wave). We have used the following notation in Fig. 1: I is an impactor, B is a buffer, CA is the contact area, FLW is the quasi-longitudinal wave front, FTRW is the quasi-transverse wave front, and FTMW is the temperature wave front. Behind the boundary of the contact area after the wavefront surface the unknown characteristics of the interaction are represented in the form of the ray series[6]

$$Z(r,t) = \sum_{k=0}^{\infty}\frac{1}{k!}[Z_{,k}]_{t=r/G}\left(t - \frac{r-r_0}{G}\right)^k H\left(t - \frac{r-r_0}{G}\right) \quad (2.1)$$

where $[Z_{,k}] = Z^+_{,k} - Z^-_{,k} = [\partial^k Z/\partial t^k]$ are the jumps in the k-th order derivatives in the time t of the required function Z on the wave surface Σ, i.e., when $t = (r - r_0)/G^{(\gamma)}$, r_0 is the initial radius, the superscripts + and − denote the quantity calculated directly in front and behind the wave front respectively, $H(t)$ is the Heaviside unit function and r is the length of the arc measured along the ray.

When using polynomial ray expansions to change from derivatives in the surface coordinate to the next order derivative in time, it is necessary to use the compatibility condition[7] on the wave surface Σ

$$G[\partial Z_{,k}/\partial r] = -[Z_{,k+1}] + \delta_t[Z_{,k}] \quad (2.2)$$

where $\delta_t = \delta/\delta t$ is the δ-derivative in time.

Differentiating in time the equations of motion (1.1) and (1.2) k times, changing from the unknown quantities to their jumps and using the compatibility condition (2.2), we obtain a system of recurrence differential equations for determining the velocities of the waves and the jumps in the unknown quantities

$$\left(1-\frac{\rho G^2}{B_r}\right)\omega_{\varphi(k+1)} = 2\delta_t\omega_{\varphi(k)} + Gr^{-1}\omega_{\varphi(k)} + b_rGX_{w(k)} - a_TGE_{(k)} + F_{\varphi(k-1)} \tag{2.3}$$

$$\left(1-\frac{\rho G^2}{KG_{rz}}\right)X_{w(k+1)} = 2\delta_tX_{w(k)} + Gr^{-1}X_{w(k)} - G\omega_{\varphi(k)} + F_{w(k-1)} \tag{2.4}$$

$$(\alpha-\tau_qG^2)E_{(k+1)} = (G^2+Gr^{-1})E_{(k)} + 2\alpha_1\delta_tE_{(k)} + F_{T(k-1)} \tag{2.5}$$

where

$$\omega_{\varphi(k)} = [\varphi_{,k+1}],\quad b_r = hK\frac{G_{rz}}{D_r},\quad X_{w(k)} = [w_{,k+1}],\quad r = r_0+Gt,\quad a_T = \frac{2(1+\nu_r)}{1-2\nu_r}\frac{\alpha\gamma}{2},$$

$$E_{(k)} = [T_{,k+1}]$$

$$F_{\varphi(k-1)} = -\delta_{tt}\omega_{\varphi(k-1)} - Gr^{-1}\delta_t\omega_{\varphi(k-1)} + G^2r^{-2}E_\theta E_r^{-1}\omega_{\varphi(k-1)} - b_rG\delta_tX_{w(k-1)}$$
$$+ b_rG^2\omega_{\varphi(k-1)} + a_TG\delta_TE_{(k-1)}$$

$$F_{w(k-1)} = -\delta_{tt}X_{w(k-1)} - Gr^{-1}\delta_tX_{w(k-1)} + G\delta_t\omega_{\varphi(k-1)} + G^2r^{-1}\omega_{\varphi(k-1)}$$

$$F_{T(k-1)} = -\alpha\delta_{tt}E_{(k-1)} - Gr^{-1}\delta_tE_{(k-1)}$$

When $k=-1$ we obtain from Eqs. (2.3)–(2.5):
for the longitudinal wave

$$G^{(1)} = \sqrt{B_r/\rho},\quad \omega^{(1)}_{\varphi(0)}\neq 0,\quad X^{(1)}_{w(0)} = 0,\quad E^{(1)}_{(0)} = 0 \tag{2.6}$$

for the transverse wave

$$G^{(2)} = \sqrt{KG_{rz}/\rho},\quad X^{(2)}_{w(0)}\neq 0,\quad \omega^{(2)}_{\varphi(0)} = 0,\quad E^{(2)}_{(0)} = 0 \tag{2.7}$$

for the temperature wave

$$G^{(3)} = \sqrt{\alpha/\tau_q},\quad E^{(3)}_{(0)}\neq 0,\quad X^{(3)}_{w(0)} = 0,\quad \omega^{(3)}_{\varphi(0)} = 0 \tag{2.8}$$

The superscript $\gamma = 1, 2, 3$ in brackets denotes the number of the wave.
For the first wave, integrating Eq. (2.3) with $k=0$, we obtain

$$\omega^{(1)}_{\varphi(0)} = c_0^{(1)}r_1^{-1/2} \tag{2.9}$$

It follows from Eq. (2.4) that

$$X^{(1)}_{w(1)} = (e_r-1)G^{(1)}c_0^{(1)}r_1^{-1/2} \tag{2.10}$$

and it follows from Eq. (2.5) that

$$E^{(1)}_{(1)} = 0 \tag{2.11}$$

Here

$$e_r = (1-KG_{rz}/B_r)^{-1} = (1-G^{(2)2}/G^{(1)2})^{-1} > 0,\quad r_\gamma = G^{(\gamma)}t+r_0,\quad \gamma = 1, 2, 3$$

and $c_0^{(1)}$ is an arbitrary constant.

To determine the jumps $\omega^{(1)}_{\varphi(1)}, X^{(1)}_{w(2)}, E^{(1)}_{(2)}$ it is necessary to substitute the known quantities $\omega^{(1)}_{\varphi(0)}, X^{(1)}_{w(1)}, E^{(1)}_{(1)}$ into system of equations (2.3)–(2.5) with $k=1$. As a result we obtain

$$\Omega^{(\gamma)}_{ij\chi} = c_i^{(\gamma)}r_\gamma^{-1/2} + \frac{1}{2}\left(\frac{E_\theta}{E_r}(2-\gamma) - \frac{1}{4} + (-1)^\gamma\chi\right)G^{(\gamma)}c_j^{(\gamma)}r_\gamma^{-3/2} + (-1)^\gamma\frac{1}{2}b_re_rG^{(\gamma)}c_j^{(\gamma)}r_\gamma^{1/2}$$

$$\omega^{(1)}_{\varphi(1)} = \Omega^{(1)}_{100},\quad X^{(1)}_{w(2)} = (e_r-1)G^{(1)}\Omega^{(1)}_{101},\quad E^{(1)}_{(2)} = 0 \tag{2.12}$$

where $c_1^{(1)}$ is an arbitrary constant.

Following the scheme described above, we determine the jumps of the next orders on the first wave with $k = 2$

$$\omega^{(1)}_{\varphi(2)} = \Omega^{(1)}_{210} + \frac{1}{8}b_r^2 e_r^2 G^{(1)2} c_0^{(1)} r_1^{3/2} - \frac{1}{8}\left(\frac{9}{4} - \frac{E_\theta}{E_r}\right)\left(\frac{E_\theta}{E_r} - \frac{1}{4}\right) G^{(1)2} c_0^{(1)} r_1^{-5/2}$$

$$X^{(1)}_{w(3)} = (e_r - 1)G^{(1)}\left[\Omega^{(1)}_{211} + (e_r - 1)\left(\frac{E_\theta}{E_r} - 1\right) G^{(1)2} c_0^{(1)} r_1^{-5/2} + \frac{1}{8}\left(\frac{E_\theta}{E_r} - \frac{1}{4}\right)^2 G^{(1)2} c_0^{(1)} r_1^{-5/2}\right.$$

$$\left. + \left(e_r - \frac{1}{4}\right) b_r e_r G^{(1)2} c_0^{(1)} r_1^{-1/2} + \frac{1}{8} b_r^2 e_r^2 G^{(1)2} c_0^{(1)} r_1^{3/2}\right]$$

$$E^{(1)}_{(3)} = 0 \tag{2.13}$$

Similarly, on the second wave we obtain

$$X^{(2)}_{w(0)} = c_0^{(2)} r_2^{-1/2}, \quad \omega^{(2)}_{\varphi(1)} = b_r e_r G^{(2)} c_0^{(2)} r_2^{-1/2}, \quad X^{(2)}_{w(1)} = \Omega^{(2)}_{100}, \quad \omega^{(2)}_{\varphi(2)} = b_r e_r G^{(2)} \Omega^{(2)}_{101}$$

$$X^{(2)}_{w(2)} = \Omega^{(2)}_{210} + \frac{1}{8} b_r^2 e_r^2 G^{(2)2} c_0^{(2)} r_2^{3/2} + \frac{9}{128} G^{(2)2} c_0^{(2)} r_2^{-5/2}$$

$$\omega^{(2)}_{\varphi(3)} = b_r e_r G^{(2)}\left\{\Omega^{(2)}_{211} + \frac{1}{8} b_r^2 e_r^2 G^{(2)2} c_0^{(2)} r_2^{3/2} - \left[e_r\left(1 - \frac{E_\theta}{E_r}\right) + \frac{15}{128}\right] G^{(2)2} c_0^{(2)} r_2^{-5/2}\right.$$

$$\left. + \left(e_r + \frac{3}{4}\right) b_r e_r G^{(2)2} c_0^{(2)} r_2^{-1/2}\right\}$$

$$E^{(2)}_{(1)} = 0, \quad E^{(2)}_{(2)} = 0, \quad E^{(2)}_{(3)} = 0 \tag{2.14}$$

where $c_0^{(2)}$ and $c_1^{(2)}$ are arbitrary constants.

On the third wave

$$E^{(3)}_{(0)} = (e^{Gr} c_0^{(3)} e_3)^{-\alpha/2}, \quad \omega^{(3)}_{\varphi(1)} = -\frac{a_T}{b_T} G^{(3)} E^{(3)}_{(0)}, \quad X^{(3)}_{w(1)} = 0$$

$$E^{(3)}_{(1)} = \frac{G^{(3)}}{2\alpha}\left(-\frac{1}{r} + \frac{G^{(3)}}{2\alpha}\ln r + \frac{G^{(3)2} r}{2\alpha}\right) E^{(3)}_{(0)} + c_1^{(3)} E^{(3)}_{(0)}$$

$$\omega^{(3)}_{\varphi(2)} = \frac{a_T}{b_T} G^{(3)} E^{(3)}_{(0)}\left[\frac{G^{(3)}}{\alpha b_r r}(1 - \alpha + G^{(3)} r) - \frac{G^{(3)2}}{4\alpha^2}(2\alpha + \ln r + G^{(3)} r) - c_1^{(3)}\right]$$

$$X^{(3)}_{w(2)} = \frac{a_T G^2}{b_T c_T} E^{(3)}_{(0)}$$

$$X^{(3)}_{w(3)} = \frac{a_T G^{(3)2}}{b_T c_T} E^{(3)}_{(0)}\left[\frac{G^{(3)}}{\alpha r}\left(\frac{1}{2} - \frac{c_T + b_r}{b_r c_T} + (G^{(3)} r - \alpha)\left(\frac{1}{b_r} - \frac{1}{c_T} - 1\right)\right) + c_1^{(3)} + \frac{G^{(3)2}}{4\alpha^2}(\ln r + G^{(3)} r)\right] \tag{2.15}$$

where

$$b_T = 1 - \frac{\rho G^{(3)2}}{B_r}, \quad c_T = 1 - \frac{\rho G^{(3)2}}{K G_{rz}}$$

and $c_0^{(3)}$ and $c_1^{(3)}$ are arbitrary constants.

After determining the unknowns we can write the bending function w, the shearing force on the boundary of the contact area Q_r and the temperature T in the form of sections of ray series, apart from integration constants:

$$w \cong \sum_{\gamma=1}^{3}\sum_{k=0}^{2}\frac{1}{k!}X_{w(k)}^{(\gamma)}y_\gamma^k H(y_\gamma)$$

$$Q_r = hKG_{rz}\left(\frac{\partial w}{\partial r}-\varphi\right) \cong KG_{rz}h\sum_{\gamma=1}^{3}\sum_{k=0}^{2}\frac{1}{k!}\left(-X_{w(k)}^{(\gamma)}G^{(\gamma)-1}+\delta_t X_{w(k-1)}^{(\gamma)}G^{(\gamma)-1}-\omega_{\varphi(k-1)}^{(\gamma)}\right)y_\gamma^k H(y_\gamma)$$

$$T \cong \sum_{\gamma=1}^{3}\sum_{k=0}^{2}\frac{1}{k!}E_{(k)}^{(\gamma)}y_\gamma^k H(y_\gamma); \quad y_\gamma = t-(r-r_0)G^{(\gamma)-1} \tag{2.16}$$

The quantities $X_{w(k)}^{(\gamma)}$, $\omega_{\varphi(k)}^{(\gamma)}$, $E_{(k)}^{(\gamma)}$ and their δ-derivatives are taken with $y_\gamma = 0$.

3. The dynamic contact problem

An elastic impactor, modelled by a load of mass m and an elastic cylindrical element with Young's modulus E_1, heated to a temperature T_1, strikes perpendicularly with velocity V_0 a target in the form of a thin plate (Fig. 1).

The motion of the impactor after contact with the target is described by the equation

$$m\ddot{y} = -P(t) \tag{3.1}$$

The equation of motion of the contact area has the form

$$\rho h\pi r_0^2\ddot{w} = 2\pi r_0 Q_r + P(t); \quad P(t) = E_1(\beta - w) \tag{3.2}$$

Here $y = \beta + w$ is the total displacement of the impactor, which is made up of the displacements of its upper and lower ends, and E_1 is the stiffness coefficient of the elastic element of the impactor.

Substituting the quantities y and $P(t)$ into Eqs. (3.1) and (3.2) and taking the condition for the tangent to the middle surface of the plate at the boundary points of the contact area to be horizontal, we arrive at a system of equations with boundary conditions which define the interaction between the impactor and target

$$m(\ddot{\beta}+\ddot{w}) = -E_1(\beta - w), \quad \rho h\pi r_0^2\ddot{w} = 2\pi r_0 Q_r + E_1(\beta - w)$$
$$r = r_0: \partial w/\partial r = 0 \tag{3.3}$$

The initial conditions for dynamic contact have the form

$$t = 0: \dot{\beta} = V, \quad \dot{w} = 0, \quad T = T_1 \tag{3.4}$$

To solve the system of equations with boundary conditions (3.3) we will represent the functions occurring in it in the form of power series in the time t. For this purpose we write the ray series (2.16) for w and Q_r on the boundary of the contact area, i.e., when $r = r_0$ and we represent the function β in the form

$$\beta = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 + \beta_5 t^5 \tag{3.5}$$

where $\beta_0, \beta_1, \ldots \beta_5$ are as yet unknown constants.

Substituting the expressions obtained into the equations with boundary conditions (3.4), taking relations (2.6)–(2.15) into account and equating coefficients of like powers of t, at each step we arrive at a system of three algebraic equations, by solving which we can obtain the graphical relations $P(t)$ and $w(t)$.

4. Numerical investigations

To illustrate the results obtained we will consider a numerical example with $m = 0.3$ kg, $r_0 = 100$ mm, $h = 200$ mm, $E_1 = 25$ kN/m, $E_\theta = E_r = 200$ GPa, $V_0 = 10$ m/s, $\nu_r = 0.3$ and $\rho = 7850$ kg/m^3 and we will investigate how the interaction force at the point of contact of the impactor and the target and the dynamic bending depend on the thermoelastic characteristic of the target a_T.

In Figs. 2 and 3 we show graphs of the time dependence of the dynamic bending and the interaction force at the point of contact of the impactor and the target for different values of a_T. The case $a_T = 0$ corresponds to the impact action, ignoring the heat spread. It can be seen that an increase in the temperature of the impactor leads to an increase in the dynamic bending and the interaction force at the contact point. One can also see that a change in a_T has a considerable effect on the dynamic bending.

By analysing the relations obtained and the behaviour of the dynamic bending and the interaction force at the point of contact of the impactor and the target we can draw the following conclusions: the temperature wave leads the elastic shock waves, the heat spread in the target first of all affects the longitudinal tension/compression wave and only in subsequent approximations does it affect the transverse shear wave, the heat spread in the target has a considerable effect on the dynamic characteristics of the contact interaction, and when the

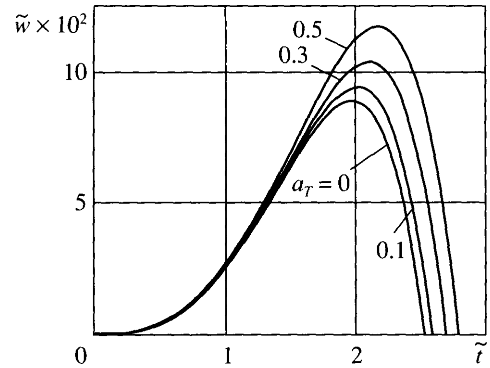

Fig. 2.

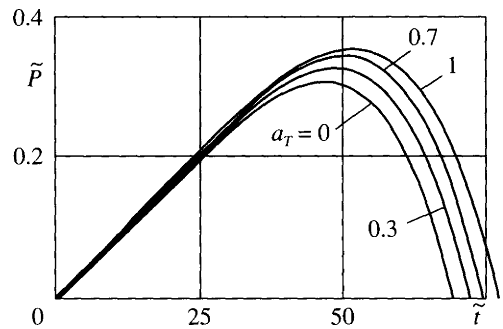

Fig. 3.

propagation of the thermoelastic wave is taken into account the dynamic bending and the interaction force at the point of contact of the impactor and the target increase, but the latter increases less intensively.

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